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B Binz†, X Zotos†‡ and D Baeriswyl†

† Institut de Physique théorique, Pérolles, CH-1700 Fribourg, Switzerland
‡ Institut Romand de Recherche Numérique en Physique des Matériaux (IRRMA), EPFL-PPH, CH-1015 Lausanne, Switzerland

Abstract. The t–J model with constant t and J between any pair of sites is studied by exploiting the symmetry of the Hamiltonian with respect to site permutations. For a given number of electrons and a given total spin the exchange term simply yields an additive constant. Therefore the real problem is to diagonalize the 't model', or equivalently the infinite U Hubbard Hamiltonian. Using extensively the properties of the permutation group, we are able to find explicitly both the energy eigenvalues and eigenstates, labelled according to spin quantum numbers and Young diagrams. As a corollary we also obtain the degenerate ground states of the finite U Hubbard model with infinite-range hopping $-t > 0$.

1. Introduction

It is widely accepted that the t–J model captures the essential physics of high-temperature superconductors, at least in the normal state [1]. The model is defined by the Hamiltonian

$$H = H_t + H_J \tag{1}$$

where

$$H_t = -\sum_{i,j,\sigma} t_{ij} P c_{i\sigma}^\dagger c_{j\sigma} P \tag{2}$$

describes the hopping between sites i and j, and

$$H_J = \sum_{i,j} J_{ij}(\boldsymbol{S}_i \cdot \boldsymbol{S}_j - \tfrac{1}{4} n_i n_j) \tag{3}$$

is the exchange interaction. The operators $c_{i\sigma}^\dagger$ ($c_{i\sigma}$) create (destroy) electrons at site i with spin σ, P is a projection operator on the subspace with no doubly occupied sites, $\boldsymbol{S}_i$ is the spin operator and n_i the particle density restricted to the values 0 and 1.

Usually both the hopping terms t_{ij} and exchange interactions J_{ij} are chosen to be non-zero if i and j are nearest neighbours and zero otherwise. Unfortunately, the model is then very difficult to solve, and explicit analytical results have so far only been obtained for a one-dimensional chain, and even then only for specific values of nearest-neighbour couplings, namely $J = 2t$ [2, 3] and $J = 0$ [4].

In this paper we consider the rather artificial model with couplings of unlimited range, i.e. $t_{ij} = t$, $J_{ij} = J$ for all sites i,j. Note that the exchange term is then simply given by $J[S(S + 1) - N^2/4]$, where N is the number of particles. Thus the real problem is to solve the 't model', which is equivalent to the infinite U Hubbard model. Its ground state has been derived previously [5]. A general solution has been conjectured by Li and

Mattis [6], on the basis of spectra obtained by exact diagonalization. Very recently, Kirson [7], exploiting the supersymmetry of the model, has calculated analytically both the energy spectrum and the degeneracies. One of us (BB) has independently solved the model using extensively the properties of the permutation group [8]. This method, described in detail below, not only offers an alternative method for deriving the energy eigenvalues but also yields all the eigenstates explicitly.

The paper is organized as follows. Section 2 summarizes the general properties of the permutation group and its irreducible representations. The Young symmetrizers allow us to decompose the Hilbert space of many-electron states into subspaces which transform according to the irreducible representations of the permutation group. In section 3 these symmetrized states are constructed explicitly and characterized by Young tableaux where the numbers of sites are replaced by symbols indicating the occupancy of the sites, i.e. $0, \uparrow, \downarrow$. In section 4 the Hamiltonian is diagonalized for the subspaces belonging to the different irreducible representations (or Young diagrams). In section 5 the technique is extended to the case of the Hubbard model (with hopping of unlimited range). This case is in general more complicated, but for $t < 0$ and $U > 0$ the exact ground state can be given. Certain mathematical details are treated in two appendices.

2. The permutation symmetry and its implications

The Hilbert space of quantum states $\mathcal{H}$ is generated by the Fock states

$$|\phi\rangle = c^{\dagger}_{i_1\uparrow} \ldots c^{\dagger}_{i_u\uparrow} c^{\dagger}_{j_1\downarrow} \ldots c^{\dagger}_{j_d\downarrow} |0\rangle \qquad (4)$$

u and d being the number of $\uparrow$ and $\downarrow$ spins, respectively. We consider these two quantities as arbitrary but fixed. We therefore specify the number of electrons $N = u + d$ and the z-component of the total spin $S^z = (u - d)/2$. The number of empty sites h (called holes) is also conserved with value $h = L - N$, as doubly occupied states have been excluded.

Since there is a constant hopping amplitude between every pair of sites, the Hamiltonian is invariant with respect to every permutation of the lattice sites. The action of such a permutation $\pi \in S_L$ on a Fock state (4) is given by the unitary operator $\rho(\pi)$ defined as follows,

$$\rho(\pi)c^{\dagger}_{i_1\sigma_1} \ldots c^{\dagger}_{i_N\sigma_N}|0\rangle := c^{\dagger}_{\pi(i_1)\sigma_1} \ldots c^{\dagger}_{\pi(i_N)\sigma_N}|0\rangle. \qquad (5)$$

Note that a transposition of two sites occupied by electrons with the same spin changes the sign of the state whereas the transposition of empty states leaves it unchanged.

Our approach is based on the commutation relations

$$[H_t, \rho(\pi)] = 0, \quad [\mathbf{S}, \rho(\pi)] = 0 \text{ and } [H_t, \mathbf{S}] = 0 \qquad \forall \pi \in S_L.$$

They allow us to label each energy level by its total spin quantum number S and a Young diagram representing the permutation symmetry. To formulate this more clearly, let us first state some facts of the representation theory of the symmetric group developed at different levels in [9–11].

There is a one-to-one correspondence between the irreducible representations of S_L and the partitions α of L (i.e. the lists $\alpha = (\alpha_1, \alpha_2, \ldots)$ of integers with the constraints $\alpha_1 + \alpha_2 + \cdots = L$ and $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant 0$).

The partitions and the corresponding irreducible representations are usually visualized in terms of a *Young diagram* noted [α].

$$[\alpha] = \begin{array}{l} \text{(diagram)} \end{array} \quad \begin{array}{l} \alpha_1 \text{ boxes} \\ \alpha_2 \text{ boxes} \end{array}$$

We enumerate the boxes of the diagram column by column from top to bottom to obtain a *Young tableau* t^α. For example, here is the tableau of the diagram $[3, 2, 1^2]$:

$$t^{(3,2,1^2)} = \begin{array}{|c|c|c|} \hline 1 & 5 & 7 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}$$

This tableau defines by its rows the dissection of $\{1, \ldots, 7\}$ into subsets $\{1, 5, 7\}$, $\{2, 6\}$, $\{3\}$ and $\{4\}$, while it defines by its columns the dissection of $\{1, \ldots, 7\}$ into subsets $\{1, 2, 3, 4\}$, $\{5, 6\}$ and $\{7\}$. Correspondingly we associate with the rows of $t^{(3,2,1^2)}$ the subgroup $R^{(3,2,1^2)} = S_{\{1,5,7\}} \times S_{\{2,6\}}$ of S_L, called horizontal group or group of the row permutations, while we obtain from the columns the subgroup $C^{(3,2,1^2)} = S_{\{1,2,3,4\}} \times S_{\{5,6\}}$ of S_L, called vertical group or group of the column permutations. The generalization to arbitrary tableaux is obvious.

We can now define the row symmetrizer $\mathcal{R}^\alpha := \sum_{\pi \in R^\alpha} \rho(\pi)$ as well as the column antisymmetrizer $\mathcal{C}^\alpha := \sum_{\pi \in C^\alpha} \mathrm{sign}(\pi)\rho(\pi)$. Finally the Young symmetrizer is given by the product of the two: $e^\alpha := \mathcal{C}^\alpha \mathcal{R}^\alpha$. The Young symmetrizers provide our main working tool for finding the eigenstates of the model. The following results of the theory are crucial [9].

Proposition 1. If $|\phi\rangle$ is an arbitrary element of $\mathcal{H}$ then $e^\alpha|\phi\rangle$, if not null, transforms under S_L according to the irreducible representaton [α].

Proposition 2. For a given Young symmetrizer e^α the set of symmetrized wavefunctions $e^\alpha|\phi\rangle$, $|\phi\rangle$ being of the form (4), spans a subspace $e^\alpha\mathcal{H}$ with a dimension equal to the number n_α of components [α] contained in ρ.

Let $|\Psi_i\rangle$, $i = 1, \ldots, n_\alpha$ be an orthonormal basis of $e^\alpha\mathcal{H}$, then the space spanned by the vectors $\rho(\pi)|\Psi_i\rangle$, $\pi \in S_L$ (i fixed) is a representation space for the representation [α] and one obtains n_α mutually orthogonal representation spaces according to the n_α basis vectors.

The problem of diagonalizing H_t and $\boldsymbol{S}^2$ in $\mathcal{H}$ is therefore completely solved once we have diagonalized it in each of the subspaces $e^\alpha\mathcal{H}$. As a final remark let us state that one also has the choice of interchanging the two factors in the definition of e^α in order to obtain $\tilde{e}^\alpha = \mathcal{R}^\alpha\mathcal{C}^\alpha$. Propositions 1 and 2 are true for $\tilde{e}^\alpha$ as well as for e^α and we are free to work with either of them.

3. Construction of symmetrized states

We will now apply the Young symmetrizer of a given tableau to the different Fock states in order to obtain symmetrized wavefunctions $e^\alpha|\phi\rangle$. Such a state is best represented graphically in terms of the corresponding tableau, where we replace the number $i \in \{1, \ldots, L\}$ of each box by the occupancy $\uparrow$, $\downarrow$ or 0 (empty) of the lattice site i in $|\phi\rangle$.

For example in a system of $L = 7$ sites with two up and two down spins the tableau $t^{(3,2,1^2)}$ yields the following symmetrized wavefunctions:

$$\begin{array}{|c|c|c|}\hline 0 & \uparrow & \downarrow \\\hline 0 & \downarrow \\\cline{1-2} 0 \\\cline{1-1} \uparrow \\\cline{1-1}\end{array} \quad = e^{(3,2,1^2)}c_{4\uparrow}^{\dagger}c_{5\uparrow}^{\dagger}c_{6\downarrow}^{\dagger}c_{7\downarrow}^{\dagger}|0\rangle$$

$$\begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\\hline \uparrow & \uparrow \\\cline{1-2} \downarrow \\\cline{1-1} \downarrow \\\cline{1-1}\end{array} \quad = e^{(3,2,1^2)}c_{2\uparrow}^{\dagger}c_{6\uparrow}^{\dagger}c_{3\downarrow}^{\dagger}c_{4\downarrow}^{\dagger}|0\rangle$$

$$\vdots$$

(6)

The question is now: how many (and which) of the $\frac{7!}{2!\cdot2!\cdot3!} = 210$ states given above are linearly independent?

Indeed, there is a way to answer this question without performing explicit calculations. First, according to the definition of e^{α}, for two configurations wich differ only by a row permutation the results of the symmetrization are identical ($|\phi\rangle = \rho(\pi)|\phi'\rangle$, $\pi \in R^{\alpha} \implies e^{\alpha}|\phi\rangle = e^{\alpha}|\phi'\rangle$). This implies that $e^{\alpha}|\phi\rangle$ is zero whenever two equally oriented spins are in the same row of the corresponding tableau, as in the second row of (6). This observation can be converted into a graphical rule that eliminates vanishing or linearly dependent states: choose an order for the three symbols $\uparrow, \downarrow, 0$ e.g. $0 < \uparrow < \downarrow$ and only take into account the graphs whose rows are filled in non-decreasing order; in addition make sure that there are no repeated $\uparrow$ or $\downarrow$ symbols in the rows. If we had worked with $\tilde{e}^{\alpha}$ instead of e^{α}, we would find another rule, which this time involves the columns of a tableau instead of the rows and the holes instead of the spins; the rule states that there cannot be two holes in the same column.

It then seems natural (although not immediately obvious) to merge these two rules into a single statement.

Proposition 3. A basis of the subspace $e^{\alpha}\mathcal{H}$ is given by the symmetrized wavefunctions $e^{\alpha}|\phi\rangle$ whose graphical representations obey the following conditions.

(i) The rows from left to right and the columns from top to bottom are filled in a non-decreasing order with the symbols $0 < \uparrow < \downarrow$.

(ii) No two equally oriented spins are in the same row.

(iii) No two holes are in the same column.

As a corrollary, the multiplicity n_{α} of the irreducible representation $[\alpha]$ in ρ equals the number of admissible ways of filling the diagram $[\alpha]$ with the symbols $\underbrace{0, \ldots, 0}_{h\times}, \underbrace{\uparrow, \ldots, \uparrow}_{u\times}, \underbrace{\downarrow, \ldots, \downarrow}_{d\times}$ according to these conditions.

Although proposition 3 is simple, reflecting in a natural way the fermionic nature of electrons and the bosonic nature of holes, it is not easy to prove it directly. It can nevertheless be seen to be a special case of the Littlewood–Richardson rule, as explained in appendix A.
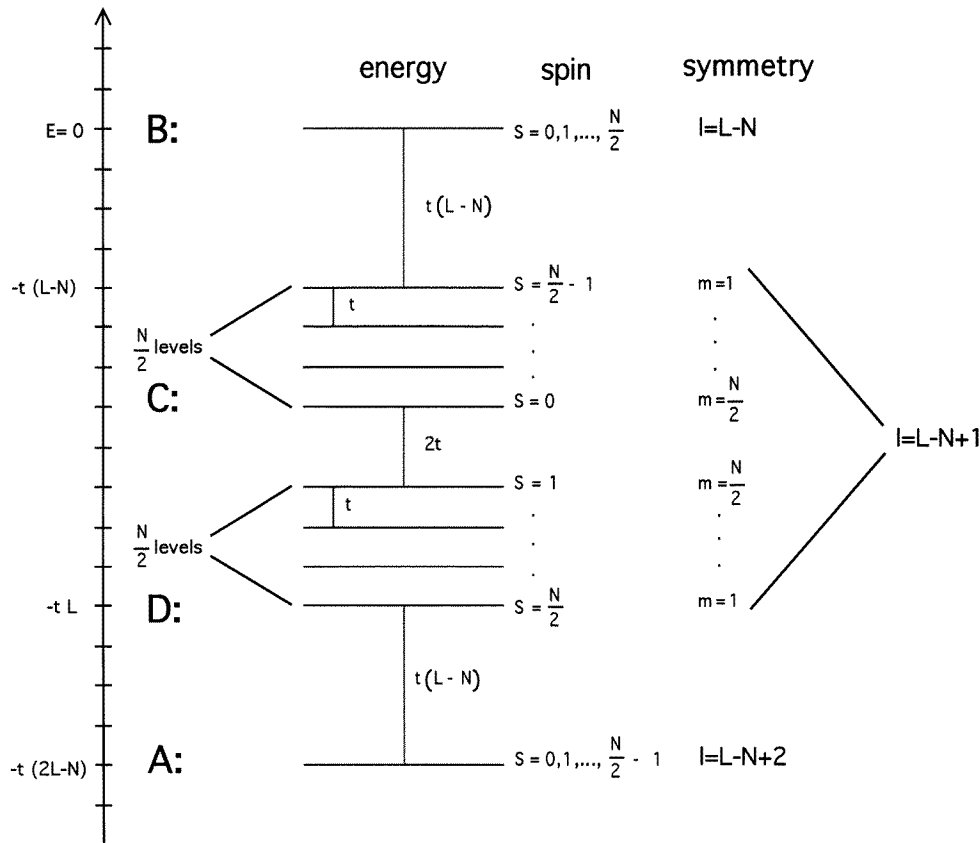
4. Spectrum and eigenstates of the model

A simple example of diagonalization using the Young symmetrizers is the single-particle problem. For $N = 1$ and $L - 1$ empty sites one can build two distinct tableaux:

$$\boxed{0}\cdots\boxed{0}\boxed{\uparrow} \qquad \begin{array}{l}\boxed{0}\cdots\boxed{0}\\\boxed{\uparrow}\end{array} \ .$$

The former corresponds to the non-degenerate eigenstate $c_{0\uparrow}^{+}|0\rangle := \sum_i c_{i\uparrow}^{+}|0\rangle$ with energy $E = -Lt$ and the latter to the eigenstate $(c_{2\uparrow}^{+} - c_{1\uparrow}^{+})|0\rangle$ which is $L - 1$ fold degenerate with energy $E = 0$.

A closer view of proposition 3 shows that the most general allowed diagram in the many-particle problem is of the form $[\alpha] = [l, 2^{m-1}, 1^{k-m}]$ with the associated tableau:

$$t^{\alpha} =$$

(7)

The width l of the first row is restricted to the values $L - N$, $L - N + 1$ and $L - N + 2$, whereas the allowed k and m values depend on S^z (the numbers of $\uparrow$ and $\downarrow$ electrons).

Once the relevant irreducible representations are specified, one can diagonalize H_t within the subspaces $e^{\alpha}\mathcal{H}$. For this purpose it is extremely convenient that H_t can be expressed in terms of permutation operators. One finds

$$H_t = -t\left[\sum_{i<j} \rho((ij)) + S^2 + f(L, N)\right]$$

with $f(L, N) = N^2/4 - (L - N)(L - N - 1)/2$ and (ij) the transposition of sites i and j.

The energy of an eigenstate is thus completely determined by its symmetry $[\alpha]$ and its total spin S. With the aid of the algebraic lemma

$$\sum_{i<j} \rho((ij))e^{\alpha} = (\ \#\ \text{transpositions in } R^{\alpha} - \ \#\ \text{transpositions in } C^{\alpha})\ e^{\alpha} \tag{8}$$

(proved in appendix B), we compute the energy as a function of S and $[\alpha] = [l, 2^{m-1}, 1^{k-m}]$. In this way we obtain the complete spectrum of the Hamiltonian H_t,

$$E(S, \alpha) = -t\left[\binom{l}{2} - \binom{k}{2} - \binom{m-1}{2} + \binom{\frac{N}{2} + S + 1}{2} + \binom{\frac{N}{2} - S}{2} - \binom{L - N}{2}\right]. \tag{9}$$

The spectrum is shown in figure 1 (2) for an even (odd) number of electrons $1 < N < L - 1$ (the case $N = L - 1$ with only one hole is treated separately). We have assumed that

Figure 1. Spectrum for even number of electrons N.

t is positive. In the opposite case the spectrum is simply inverted. Apart from the energy values we also indicate the total spin S. The column on the right-hand side of the figures refers to the permutation symmetry.

Due to the large symmetry group every single energy level of this system will in general be highly degenerate. The degeneracy of a level corresponding to $[\alpha]$ and S is $(2S+1)$ times the degree f^α of the irreducible representation $[\alpha]$. The latter can be calculated following [10] or [11] and in our case amounts to

$$f^\alpha = \frac{L!}{k!(m-1)!(l-2)!} \cdot \frac{k-m+1}{(k+l-1)(m+l-2)}. \tag{10}$$

We distinguish four parts, labelled by capital letters, that we will now discuss seperately.

4.1. A: $l = L - N + 2$ and B: $l = L - N$

Consider all the Young diagrams with $l = L-N+2$ (case A) or $l = L-N$ (case B). In both cases the multiplicity is $n_\alpha = 1$ if $|S^z| \leqslant (k-m)/2$ and 0 elsewhere. To see this, we look at the tableaux in equations (11) and (12) which represent the only allowed filling according to proposition 3. One has the liberty to invert some of the ↑ spins in the $k-m$ last boxes of the first column, but not more than these. Hence the total spin is $S = (k-m)/2$. By

energy spin symmetry

B: $\quad$ $S=\dfrac{1}{2},\dfrac{3}{2},...,\dfrac{N}{2}$ $\quad$ l=L-N

E = 0

t (L - N)

-t (L-N) $\quad$ $S=\dfrac{N}{2}-1$ $\quad$ m = 1

$\dfrac{N-1}{2}$ levels $\quad$ t

C: $\quad$ $S=\dfrac{1}{2}$ $\quad$ $m=\dfrac{N-1}{2}$

2t $\quad$ l=L-N+1

$S=\dfrac{1}{2}$ $\quad$ $m=\dfrac{N+1}{2}$

$\dfrac{N+1}{2}$ levels $\quad$ t

D: $\quad$ $S=\dfrac{N}{2}$ $\quad$ m = 1

-t L

t (L - N)

-t (2L-N) $\quad$ A: $\quad$ $S=\dfrac{1}{2},\dfrac{3}{2},...,\dfrac{N}{2}-1$ $\quad$ l=L-N+2
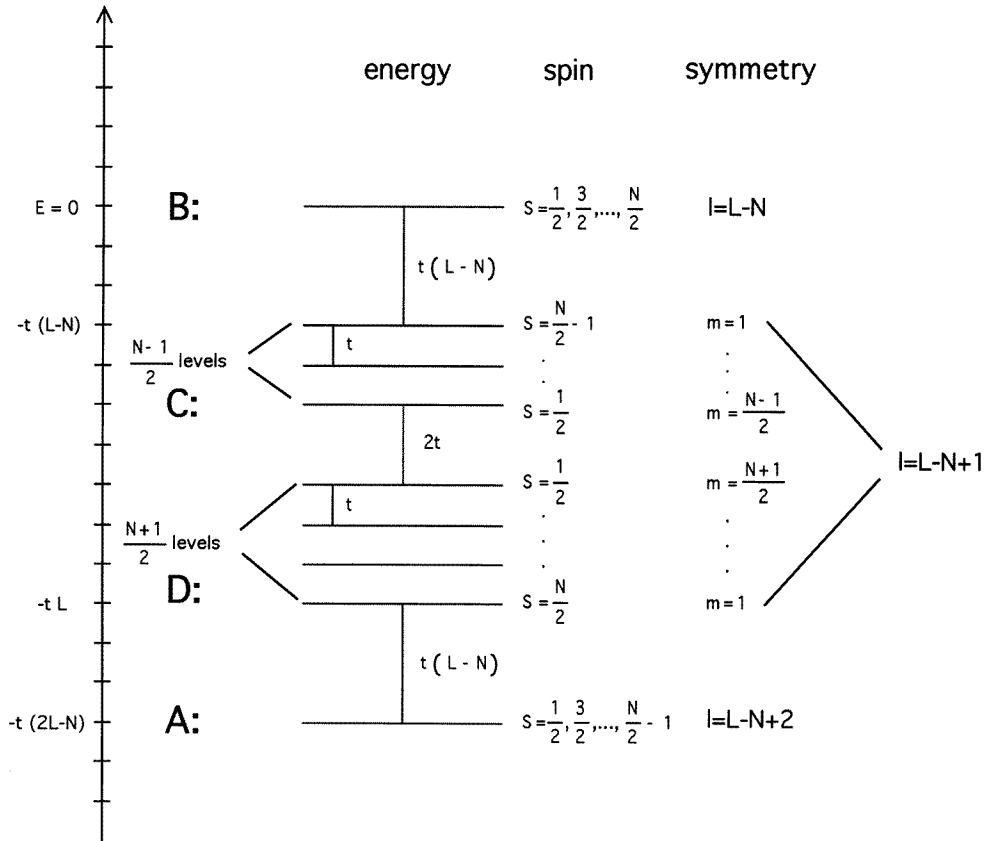
Figure 2. Spectrum for odd number of electrons N.

varying k and m with l and L fixed one obtains every possible value for S in case B and every value exept the completely magnetized $S = N/2$ in case A.

The energies given by (9) turn out to be $E_A = -t(2L-N)$ (resp. $E_B = 0$) independently of the different values of S, which leads to an accidental degeneracy. This means that there are states of different symmetries and spin values with the same energy. This degeneracy is lifted by the term H_J in the Hamiltonian (1).

Because the dimension of $e^\alpha \mathcal{H}$ is one, the state $e^\alpha |\phi\rangle$ is an eigenstate of H_t. In the case A ($l = L - n + 2$), it is convenient to use $\tilde{e}^\alpha$ instead of e^α and to change the order convention of proposition 3 into $\uparrow < \downarrow < 0$. We then obtain the eigenstates A:

$$\begin{array}{|c|c|c|c|c|} \hline \uparrow & \downarrow & 0 & \cdots & 0 \\ \hline \uparrow & \downarrow \\ \cline{1-2} \vdots & \vdots \\ \cline{1-2} \uparrow & \downarrow \\ \cline{1-2} \uparrow \\ \cline{1-1} \vdots \\ \cline{1-1} \uparrow \\ \cline{1-1} \end{array} \quad = b_{2\,k+2}^\dagger \ldots b_{m\,k+m}^\dagger c_{m+1\uparrow}^\dagger \ldots c_{k\uparrow}^\dagger \sum_{i<j\in\Lambda} b_{ij}^\dagger |0\rangle$$

(11)

where $b_{ij}^\dagger := c_{i\uparrow}^\dagger c_{j\downarrow}^\dagger + c_{j\uparrow}^\dagger c_{i\downarrow}^\dagger$ creates a singlet pair on the sites i and j and Λ is the sublattice

formed by the l sites in the first row of t^α. This is only one particular eigenstate of this level. In fact, as already stated in [5], a general eigenstate of level A with total spin S is of the form

$$P\psi^\dagger_{(N-2,S)}c^+_{0\uparrow}c^+_{0\downarrow}|0\rangle$$

where $\psi^\dagger_{(N-2,S)}$ is an arbitrary wavefunction of $N-2$ electrons with spin S, $c^+_{0\sigma}:=\sum_i c^+_{i\sigma}$ creates an electron in the single-particle ground state and P projects out states with doubly occupied sites.

An eigenstate of level B is given by

$$=\sum_{\pi\in S_k}\sum_{\tau\in S_m}\text{sign}(\pi\tau)b^\dagger_{\pi(2)\tau(k+2)}\cdots b^\dagger_{\pi(m)\tau(k+m)}c^\dagger_{\pi(m+1)\uparrow}\cdots c^\dagger_{\pi(k)\uparrow}|0\rangle$$

(12)

S_k being the set of permutations of $\{1,\ldots,k\}$ (the first column) and S_m the permutations of $\{k+1,\ldots,k+m\}$ (second column). Again this is only one representative member of a large subspace of degenerate eigenstates. The others can in principle be calculated through the repeated application of permutation operators, the spin-lowering operator $S^-:=S^x-iS^y$ and linear combinations of them. Whether there is a more compact characterization of these subspaces, as in case A, is an open question.

4.2. C and D: $l=L-N+1$

The diagrams with $l=L-N+1$ appear with multiplicity 2 if $|S^z|\leqslant(k-m-1)/2$ and with multiplicity 1 if $|S^z|=(k-m+1)/2$. The diagonalization of H_t in $e^\alpha\mathcal{H}$ leads therefore to two levels with total spin $S=(k-m\pm1)/2$. $e^\alpha\mathcal{H}$ is spanned by the two symmetrized wavefunctions:

The odd combination of them

$$|\Psi_2\rangle-|\Psi_1\rangle=\sum_{\pi\in S_k}\sum_{\tau\in S_m}\text{sign}(\pi\tau)b^\dagger_{\pi(2)\tau(k+2)}\cdots b^\dagger_{\pi(m)\tau(k+m)}$$

$$\times c^\dagger_{\pi(m+1)\uparrow}\cdots c^\dagger_{\pi(k-1)\uparrow}\left(b^\dagger_{\pi(k)\tau(k+1)}+\sum_{\nu=k+m+1}^L b^\dagger_{\pi(k)\,\nu}\right)|0\rangle$$

with $S^z=(k-m-1)/2$ is easily seen to be an eigenstate of S^2 with $S=(k-m-1)/2$ because it is annulled by the raising operator $S^+=S^x+iS^y$. Hence it has to be an eigenstate

of H_t as well. The states of this type give rise to part C of the spectrum with energies $E_C = -t(L - N/2 - 1 - S)$.

The second eigenvector with $S = (k - m + 1)/2$ must be orthogonal to $|\Psi_2\rangle - |\psi_1\rangle$ and therefore is given by the sum

$$|\Psi_2\rangle + |\Psi_1\rangle = \sum_{\pi \in S_k} \sum_{\tau \in S_m} \text{sign}(\pi\tau) b^\dagger_{\pi(2)\tau(k+2)} \dots b^\dagger_{\pi(m)\tau(k+m)}$$

$$\times c^\dagger_{\pi(m+1)\uparrow} \dots c^\dagger_{\pi(k-1)\uparrow} \left(d^\dagger_{\pi(k)\pi(1)} + d^\dagger_{\pi(k)\tau(k+1)} + \sum_{\nu=k+m+1}^{L} d^\dagger_{\pi(k)\nu} \right) |0\rangle$$

with $d^\dagger_{ij} := c^\dagger_{i\uparrow} c^\dagger_{j\downarrow} - c^\dagger_{j\uparrow} c^\dagger_{i\downarrow}$. These states correspond to part D of the spectrum and the energies are $E_D = -t(L - N/2 + S)$.

4.3. The special case of one single hole ($N = L - 1$)

For $N = L - 1$ figures 1 and 2 remain valid exept that level B now contains only the ferromagnetic states with $S = N/2$. To see this, note that the symmetrized states of level B correspond to a Young diagram with $l = L - N$. If only one hole is present, this means that $l = 1$ and thus the only remaining diagram is $[1^L]$. This diagram corresponds to the ferromagnetic state

$$\sum_{i=1}^{L} (-1)^i c^\dagger_{1\uparrow} \dots \widehat{c^\dagger_{i\uparrow}} \dots c^\dagger_{L\uparrow} |0\rangle.$$

The only degeneracy is in this case the trivial spin degeneracy $2S + 1$.

It follows that for $N = L - 1$ and a positive hopping parameter ($-t > 0$) the ground state is ferromagnetic. This result is not surprising since it is a consequence of two well known theorems, both confirming a unique ferromagnetic ground state for this particular case. The first is Tasaki's extension of Nagaoka's theorem [12] and the second is a theorem proven by Mielke on flat band ferromagnetism [13, 14].

In this model, we find an example of Nagaoka ferromagnetism where the one-hole condition is absolutely necessary, as we find always a complete spin degeneracy for $N < L - 1$.

4.4. Permutation symmetry and supersymmetry

At this point it is worthwhile to connect the present approach with that of Kirson [7], who exploited the dynamical supersymmetry of the model. His classification of many-electron states in Fock space $\mathcal{F}$ is based on the irreducible representations $[Y, S]$ of a certain superalgebra. The representation space of $[Y, S]$ contains four possible pairs of quantum numbers, namely (Y, S), $(Y + \frac{1}{2}, S - \frac{1}{2})$, $(Y - \frac{1}{2}, S - \frac{1}{2})$ and $(Y, S - 1)$, where $Y = L - \frac{1}{2}N$ and S is the total spin. We can identify the representation space of $[Y, S]$ with $e^\alpha \mathcal{F}$, where $[\alpha] = [l, 2^{m-1}, 1^{k-m}]$ is related to Y and S by $k = L - Y + S$, $m = L - Y - S + 1$ and $l = -L + 2Y + 1$.

For a given α there are (in general) four classes of symmetrized states in Fock space. These correspond to parts A–D of the spectrum with differing numbers of particles (N and $N \pm 1$), and can be identified with the four pairs of quantum numbers in $[Y, S]$:

A $\Leftrightarrow (Y - \frac{1}{2}, S - \frac{1}{2})$ B $\Leftrightarrow (Y + \frac{1}{2}, S - \frac{1}{2})$ C $\Leftrightarrow (Y, S-1)$ D $\Leftrightarrow (Y, S)$.

5. Ground states of the finite U Hubbard model with infinite range hopping in the case $-t > 0$

The method developed in the previous sections can be generalized for other models which are invariant under permutations of the lattice sites. For instance the Hubbard model with infinite-range hopping,

$$H_{\text{Hubb}} = H_0 + H_U$$
$$H_0 = -t \sum_{i,j,\sigma} c_{i\sigma}^\dagger c_{j\sigma}$$
$$H_U = U \sum_i n_{i\uparrow} n_{i\downarrow}$$

can in principle be treated in the same way. This model has been considered by several autors [14, 15] but a general solution is still lacking. The most important new feature is the appearance of doubly occupied sites. Proposition 3 has to be modified in a way so as to treat these sites as well. The procedure is the following.

(i) Compute the admissible tableaux (the basis states of $e^\alpha \mathcal{H}$) without double occupation as explained in section 3 or appendix A.

(ii) Replace a pair $\uparrow, \downarrow$ of symbols by $\uparrow\downarrow, 0$ and compute the symmetrized states with exactly one double occupation. The new symbol $\uparrow\downarrow$ has to be included in the ordering convention, e.g. $0 < \uparrow < \downarrow < \uparrow\downarrow$.

(iii) Replace another two symbols $\uparrow, \downarrow$ by $\uparrow\downarrow, 0$ and continue, until there is no pair $\uparrow, \downarrow$ left. In proposition 3, the symbols $\uparrow\downarrow$ are treated like the holes, i.e. they must not be repeated within the same column.

A model which includes doubly occupied sites is much more difficult to solve than the model considered in this paper. Nevertheless there is a particular class of diagrams where these complications do not matter.

Consider a diagram of the form (7) where the number l of boxes in the first row equals $L - N$. In the tableaux of this kind, there is no way to produce a doubly occupied state without violating the rules, because there is no room for an additional hole. The only symmetrized states according to such a diagram are therefore the states B, eigenstates of H_t, which contain no double occupation. We conclude that every eigenstate of H_t belonging to case B is at the same time an eigenstate of the Hubbard Hamiltonian with the unchanged energy $E_B = 0$.

Since in the case $-t > 0$ and $U > 0$ we find $\langle \psi | H_0 | \psi \rangle \geqslant 0$ and $\langle \psi | H_U | \psi \rangle \geqslant 0$ for every state $|\psi\rangle$, the states B are even the (only) ground states of H_{Hubb}. It is remarkable that the term H_J splits the accidental degeneracy of level B, while this degeneracy remains exact in the Hubbard model for every positive value of U. This shows that the $t-J$ model does not capture correctly the behaviour of the Hubbard model, not even in the asymptotic region $U >> |t|$. (In fact, a systematic large U expansion of the Hubbard model yields, in addition to the exchange term, another contribution, the so-called pair-hopping term. See e.g. [16].)

6. Conclusion

We have shown that the permutation symmetry of the $t-J$ model with infinite-range hopping allows us to derive explicitly the energy spectrum, the eigenfunctions and their quantum numbers. The model is admittedly rather unphysical due to the complete lattice connectivity which leads to unusually high level degeneracies. Nevertheless the many-body spectrum

has a very rich structure, and therefore the model deserves to be added to the few non-trivial cases of exactly solvable strongly correlated fermion systems. Our results for the spectrum and the degeneracies agree with those derived on the basis of a dynamical supersymmetry [7], but in addition we have also been able to obtain all the eigenstates. Furthermore, we have found an exact correspondence between the two approaches.

Acknowledgments

Appendix A. The Littlewood–Richardson rule and proof of proposition 3

Sometimes a representation of a group G is completely determined by a representation of a subgroup of G. To formulate this properly, we refer to the concept of induced representations.

Proposition A.1. Given a subgroup H of a finite group G and a representation σ of H, there always exists a representation ρ of G into a vector space V and a subrepresentation $\tilde{\sigma}$ of $\rho|_H$ into a subspace W of V such that $\tilde{\sigma}$ is equivalent to σ and

$$V = \bigoplus_{\gamma \in G/H} W_\gamma$$

where G/H is the set of left cosets of H in G and $W_\gamma = \rho(s)W$ for an arbitrary $s \in \gamma$.

In this situation, ρ is (up to an equivalency) uniquely determined by σ and is called the induction of σ into G.

Let $[\alpha]$ be an irreducible representation of S_n and $[\beta]$ an irreducible representation of S_m, then the tensor product $[\alpha] \otimes [\beta]$ yields an irreducible representation of $S_n \times S_m$. $S_n \times S_m$ can be identified in a natural way with a subgroup of S_{n+m}, if S_n acts on the elements $\{1, 2, \ldots, n\}$ and S_m acts on $\{n+1, n+2, \ldots, n+m\}$. The outer product $[\alpha][\beta]$ is defined as the induction of $[\alpha] \otimes [\beta]$ into S_{n+m} and is in general a reducible representation of S_{n+m}. This multiplication is associative, commutative and obeys a distributive law together with the direct sum $\oplus$.

The representation ρ defined in equation (5) is an outer product:

$$\rho = [h][1^u][1^d].$$

To see this, consider one particular Fock state $|\phi\rangle$ of the form (4). The subgroup of S_L that leaves $|\phi\rangle$ invariant (up to a sign) is isomorphic to $S_h \times S_u \times S_d$. The one-dimensional subspace W of $\mathcal{H}$ generated by $|\phi\rangle$ therefore carries the representation $\sigma = [h] \otimes [1^u] \otimes [1^d]$ of $S_h \times S_u \times S_d < S_L$. All we have to verify is that the Hilbert space of the system (with N and S^z fixed) is the direct sum

$$\mathcal{H} = \bigoplus_{\gamma \in S_L/S_h \times S_u \times S_d} \rho(\pi_\gamma)W$$

where $\pi_\gamma \in \gamma$ is a representative member of the left coset γ.

The Littlewood–Richardson rule describes a graphical way to generate the irreducible constituents of an arbitrary product $[\alpha][\beta]$, but for our purpose it is sufficient to consider a product of the form $[\alpha][1^n]$.

Proposition A.2. The diagrams $[\gamma]$ of the irreducible constituents of $[\alpha][1^n]$ may be calculated by adding n boxes to the diagram $[\alpha]$ in all possible ways such that no two added boxes appear in the same row.

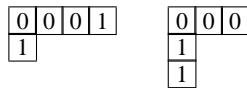*Example.* If $[\alpha] = [3, 1^2]$ and $n = 2$ we obtain
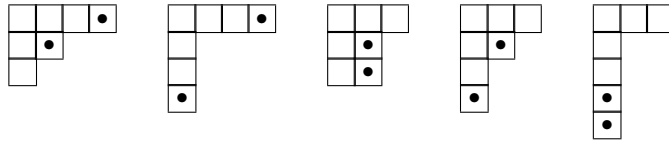
Thus

$$[3, 1^2][1^2] = [4, 2, 1] \oplus [4, 1^3] \oplus [3, 2^2] \oplus [3, 2, 1^2] \oplus [3, 1^4].$$

We can apply this process repeatedly in order to obtain the constituents of $\rho = [h][1^u][1^d]$.
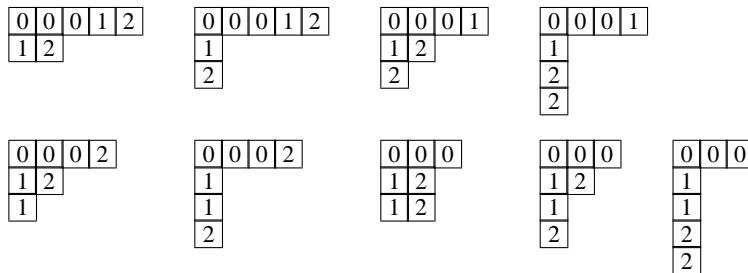
For example, if we wish to calculate $[3][1^2][1^2]$, we first evaluate $[3][1^2]$:

This yields

$$[3][1^2] = [4, 1] \oplus [3, 1^2]$$

so that the constituents of $[3][1^2][1^2]$ are obtained as follows:

Therefore

$$[3][1^2][1^2] = [5, 2] \oplus [5, 1^2] \oplus 2[4, 2, 1] \oplus 2[4, 1^3] \oplus [3, 2^2] \oplus [3, 2, 1^2] \oplus [3, 1^4].$$

This algorithm leads to the same diagrams as proposition 3, if we replace the numbers 1, (2) by the symbols $\uparrow$, ($\downarrow$) respectively.

Proposition 3 describes how to obtain a basis of the subspace $e^\alpha \mathcal{H}$ for a given diagram $[\alpha]$. The procedure described above shows only that proposition 3 leads to the correct dimension of $e^\alpha \mathcal{H}$. However, we have also seen that this dimension is never higher than two. Thus it is easy to verify in every case that the corresponding states $e^\alpha |\phi\rangle$ are linearly independent.

Appendix B. Proof of (8)

We first note, that $\sum_{i<j} \rho(ij)$ commutes with every $\rho(\pi)$, $\pi \in S_L$. Thus

$$\sum_{i<j} \rho(ij)e^\alpha = \sum_{i<j} \mathcal{C}^\alpha \rho(ij)\mathcal{R}^\alpha.$$

Clearly $\mathcal{C}^\alpha \rho(ij)\mathcal{R}^\alpha = e^\alpha$ if $(ij) \in R^\alpha$ and $\mathcal{C}^\alpha \rho(ij)\mathcal{R}^\alpha = -e^\alpha$ if $(ij) \in C^\alpha$. In the remaining case $(ij) \notin R^\alpha \cup C^\alpha$, there exists one site $k \neq i, j$, which is in the same column as i and in the same row as j.
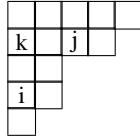
As $(ij) = (ik)(ij)(kj)$, we find

$$\mathcal{C}^\alpha \rho(ij)\mathcal{R}^\alpha = \mathcal{C}^\alpha \rho(ik)\rho(ij)\rho(kj)\mathcal{R}^\alpha = -\mathcal{C}^\alpha \rho(ij)\mathcal{R}^\alpha$$

and therefore

$$\mathcal{C}^\alpha \rho(ij)\mathcal{R}^\alpha = 0.$$

This proves equation (8).

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